SIERPIŃSKI PEDAL TRIANGLES

JIU DING, L. RICHARD HITT, AND XIN-MIN ZHANG

ABSTRACT. We generalize the construction of the ordinary Sierpiński triangle to obtain a two-parameter family of triangles we call *Sierpiński pedal triangles.* These triangles are obtained from a given triangle by recursively deleting the associated pedal triangles in a manner analogous to the construction of the ordinary Sierpiński triangle. We study the fractal dimension of these Sierpiński pedal triangles and related area ratios. We also provide some computer generated graphs of the fractals.

1. PRELIMINARIES

Let T_0 be a triangle with inner angles A_0, B_0, and C_0. The *pedal triangle* of T_0, denoted by T_1, is the triangle obtained by joining the feet of the three altitudes of T_0. Denote the inner angles of T_1 by A_1, B_1, and C_1. If T_0 is an acute triangle, then T_1 is inscribed inside T_0. If T_0 is an obtuse triangle, then two of the vertices of T_1 will fall on extensions of sides outside of T_0. If T_0 is a right triangle, then T_1 degenerates to a line segment. These cases are illustrated in Figure 1.

It is well known that when T_0 is not a right triangle, then the original triangle T_0 is similar to $\triangle A_0B_1C_1$, $\triangle A_1B_0C_1$, and $\triangle A_1B_1C_0$. We briefly outline the proof here since we are going to use this result to verify some important formulas later. We consider acute triangles only, since the same argument works for the obtuse triangles as well. From Figure 1(a) we see that to show $\triangle A_0B_1C_1$ is similar to T_0, it suffices to verify that $\angle B_0 = \angle A_0B_1C_1$. Let E be the point of intersection of the three altitudes of T_0 (called the *orthocenter*), then it is easy to see that $\angle B_0$ is the same as $\angle A_0EC_1$, and the latter equals $\angle A_0B_1C_1$ because the quadrilateral $A_0C_1EB_1$ can be inscribed in a circle in which $\angle A_0EC_1$ and $\angle A_0B_1C_1$ subtend the same arc. A similar argument applies to the similarity between T_0 and the other two triangles.

From the above similarity results, one can derive the angle and side formulas for T_1 in terms of the angles and sides of T_0 as in [Hob97, KS88]. These formulas play a key role in [HZ01] where sequences of pedal triangles are studied in detail. They are also essential in this paper. From the self-similarity property in the construction of pedal triangles, as illustrated in Figure 2, the following angle formulas can be verified immediately. If T_0 is acute, then

$$A_1 = \pi - 2A_0, \qquad B_1 = \pi - 2B_0, \qquad C_1 = \pi - 2C_0; \tag{1a}$$

$$a_1 = a_0 \cos A_0, \qquad b_1 = b_0 \cos B_0, \qquad c_1 = c_0 \cos C_0. \tag{1b}$$

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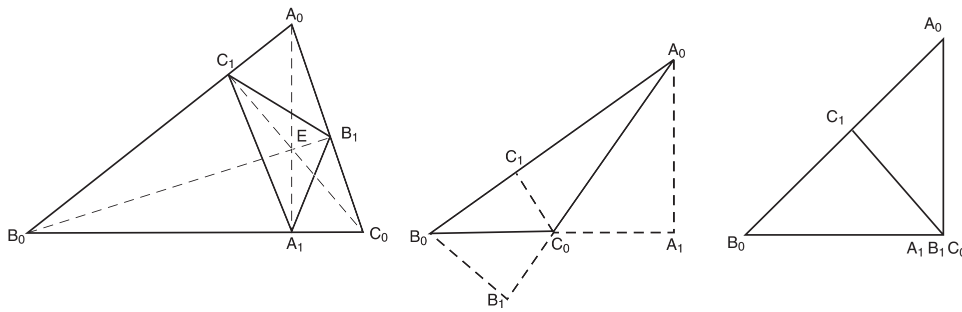

FIGURE 1. Pedal Triangle Construction

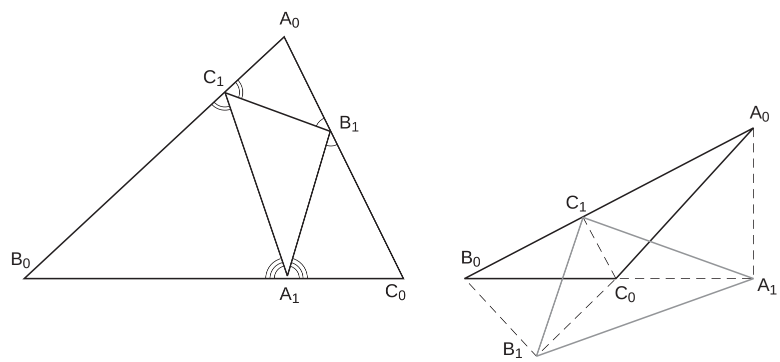

FIGURE 2. Self-Similarity Property of Pedal Triangles

where a_0, b_0, c_0, and a_1, b_1, c_1 are the sides of T_0 and T_1 respectively.

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If T_0 is obtuse with, say, $\pi > A_0 > \pi/2$, then

$$A_1 = 2A_0 - \pi, \qquad B_1 = 2B_0, \qquad C_1 = 2C_0;$$
$$a_1 = -a_0 \cos A_0, \qquad b_1 = b_0 \cos B_0, \qquad c_1 = c_0 \cos C_0.$$

For an acute triangle $\triangle A_0B_0C_0$, regarding the sides of its pedal triangle $\triangle A_1B_1C_1$ as three light beams and thinking of the sides of $\triangle A_0B_0C_0$ as three mirrors, the similarity between $\triangle A_0B_0C_0$ and each of the three smaller triangles surrounding the pedal triangle $\triangle A_1B_1C_1$ illustrates the optical property that *the angle of incidence equals the angle of reflection*. Since nature always takes the most economical way, this physical interpretation of pedal triangle also implies an interesting geometric extreme property of pedal triangle called *Fagnano's Problem*.

Theorem 1.1. [Cox89, p. 20] [CG67, p. 88] *For a given acute triangle $\triangle A_0B_0C_0$, the pedal triangle $\triangle A_1B_1C_1$ has the shortest perimeter among all triangles that are inscribed in $\triangle A_0B_0C_0$.*

The main purpose of this article is to look at pedal triangles from a different perspective, however. In the next section, we generalize the traditional construction of the Sierpiński triangle to a construction that uses pedal triangles.

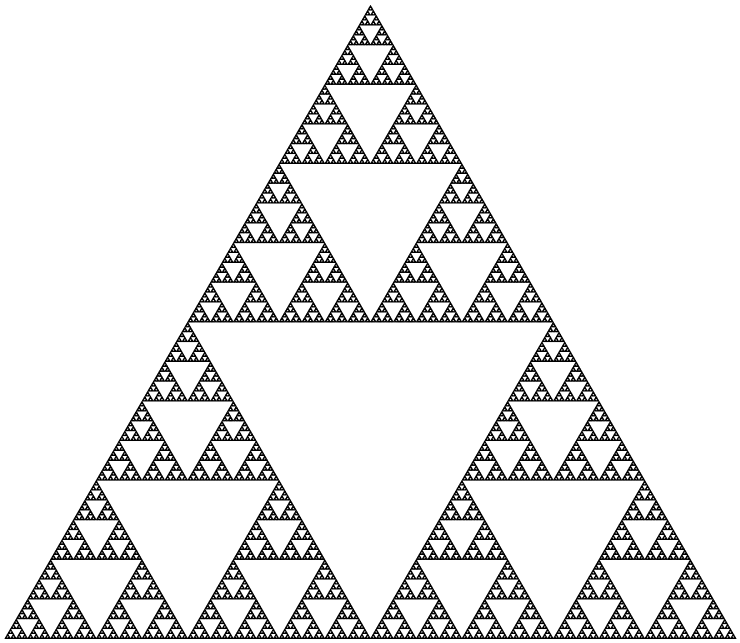

FIGURE 3. Sierpiński Triangle, 10 iterations

We call these objects *Sierpiński Pedal Triangles (SPT)*. In section 3, we compute the area ratios for these SPTs. And in section 4, we show how the fractal dimension of SPT depends on the initial triangle. Section 5 discusses the SPTs from an iterated function system perspective. And in section 6 we discuss the dynamics of the sequence of pedal triangles.

2. Sierpiński Pedal Triangle

To begin, we recall the construction of the Sierpiński triangle. Let S_0 be an equilateral triangle (or any other triangle). Joining the middle points on the three sides of S_0 results a second triangle T_0 which is similar to S_0. We call it the middle triangle of S_0. Let $S_1 = S_0 \backslash \mathrm{Int} T_0$, i.e., remove the interior of the middle triangle T_0 from S_0. Then S_1 is a union of three smaller triangles each of which is similar to S_0. From each of these smaller triangles, remove the interior of the middle triangle again, and denote the resulting union of nine equilateral triangles by S_2, $S_2 \subset S_1$. Continuing this procedure to define $S_3 \supset S_4 \supset S_5 \cdots$, one obtains the well-known *Sierpiński Triangle* (ST) by taking the intersection of the nested sequence (see Figure 3).

$$ST = \bigcap_{n=0}^{\infty} S_n.$$

The Sierpiński triangle can be viewed as one possible generalization of the Cantor set to two-dimensions. Other generalizations are possible. For example, the original triangle S_0 does not have to be equilateral. It can be any triangle, as long as one joins the middle points from each side to form a middle triangle,

mention these are equivalent via an isometry

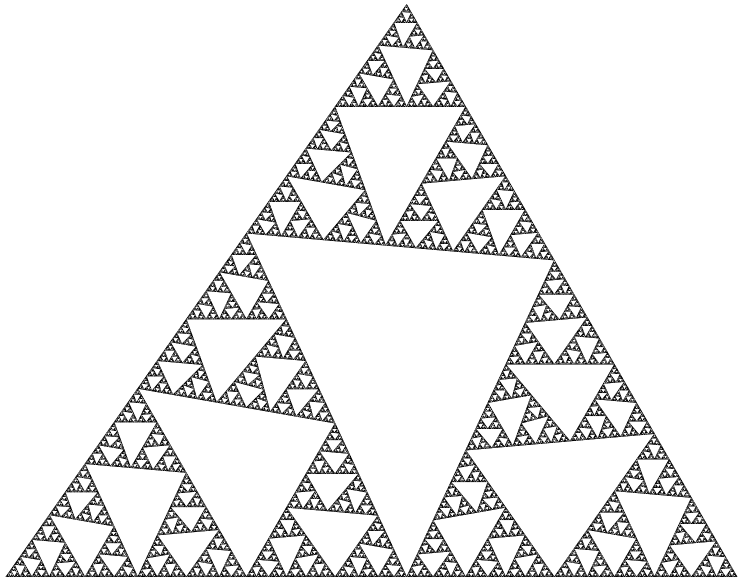

FIGURE 4. SPT constructed from (55°-60°-65°)-triangle, 12 iterations

and removes its interior from the previous triangle. The result will always be three smaller triangles that are similar to their parental triangle. Continuing this process will produce a *self-similar fractal*. It is natural to ask if these STs are the only self-similar fractal obtained by removing inscribed triangles? That is, for a given triangle T, can one remove a triangle whose vertices are on the sides of T and have the resulting three smaller triangles all similar to T? Well, as we saw in the previous section, the pedal triangle of T is such a triangle. Furthermore, it is the only such triangle other than the ordinary middle triangle. Therefore, it also generates a self-similar fractal, and we call it a *Sierpiński pedal triangle (SPT)*. Figure 4 is an example.

Introduce fractals better

improve wording

If the initial triangle T is equilateral, then the feet of the three altitudes of T coincide with the middle points of the three sides of T, hence the resulting SPT is the same as ST. In this regard, one may view SPT as a generalization of ST.

In the construction of ST, one needs only three contraction maps with the same contraction ratio $1/2$. The procedure then iterates to produce ST. In the construction of a SPT, however, one needs not only three contractions maps of (possibly) different contraction ratios, but also reflections. More importantly, the contraction ratios depend upon the shape of the initial parental triangle.

In the remainder of this paper, we examine some fundamental properties of SPT in comparison to ST and see to what extent SPT is a true generalization of ST.

Remark 2.1.

(i) ST and SPT are the only self-similar *inscribed* triangular fractals. Because the pedal triangle of an obtuse triangle is not inscribed “inside” the parental triangle, from now on, we will be concerned with only acute

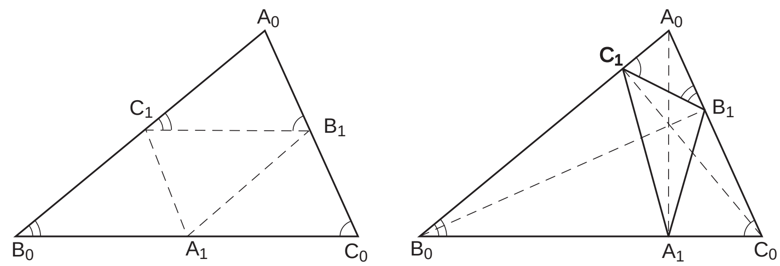

FIGURE 5. Comparison of ST and SPT Constructions

triangles. Let $\triangle ABC$ be any acute triangle and $\triangle A_1B_1C_1$ any inscribed triangle which divides the former into three smaller triangles as shown in Figure 5. If $\triangle AB_1C_1$ is similar to $\triangle ABC$, then either $\angle B = \angle AC_1B_1$ and $\angle C = \angle AB_1C_1$, or $\angle B = \angle AB_1C_1$ and $\angle C = \angle AC_1B_1$. The first case can happen only when $\triangle A_1B_1C_1$ is the middle triangle of $\triangle ABC$, and the second case occurs only for the pedal triangle.

(ii) ST and SPT are not affine equivalent. It is clear that when the initial triangle is not equilateral, SPT and ST are not affine equivalent since an affine transformation must preserve the ratio of the line segments on each side of the triangle. Moreover, SPT involves the altitudes, but orthogonality is not an affine invariant. From Figure 5 we can see that for a right triangle, its pedal triangle degenerates to a line segment while the ST is still non-degenerate.

3. The Area Ratio of A Pedal Triangle

One of the important ways to distinguish two different fractals is to compare their *fractal dimensions*. Before we consider the fractal dimension of SPT, let us look at the area ratio of a pedal triangle $\triangle A_1B_1C_1$ with its parental triangle $\triangle ABC$, and provide some additional results from the classical geometry of triangles. For the ST case, it is easy to see that the area ratio is always $1/4$ because the contraction ratio is $1/2$. Then the fractal dimension of ST is easily calculated to be $\ln 3/\ln 2$. On the other hand, for the SPT case, a direct calculation from formulas 1a and 1b shows that

$$\frac{\text{Area}\triangle A_1B_1C_1}{\text{Area}\triangle ABC} = \frac{\cos A\cos B\sin(\pi-2C)}{\sin C} = -2\cos A\cos B\cos(A+B),$$

where $0 < A, B < \pi/2$ and $\pi/2 < A+B < \pi$.

If we are concerned with only acute triangles, we may introduce the following *index domain*, $I = \{(x,y) \mid 0 < x < \pi/2,\ 0 < y < \pi/2,\ \pi/2 < x+y\}$. In Figure 6, the index domain I is the interior of the center shaded triangle M_0. The interiors of the three triangles M_1, M_2, and M_3 surrounding M_0 represent all possible obtuse triangles. The boundaries between M_0 and the M_i's, $i = 1, 2, 3$, represent all right triangles. For each $(x,y) \in I$, set $z = \pi - (x+y)$. Then the ordered triple of real numbers (x,y,z) represents an acute triangle in the Euclidean plane with inner angles x, y and z.

We will need the following definition.

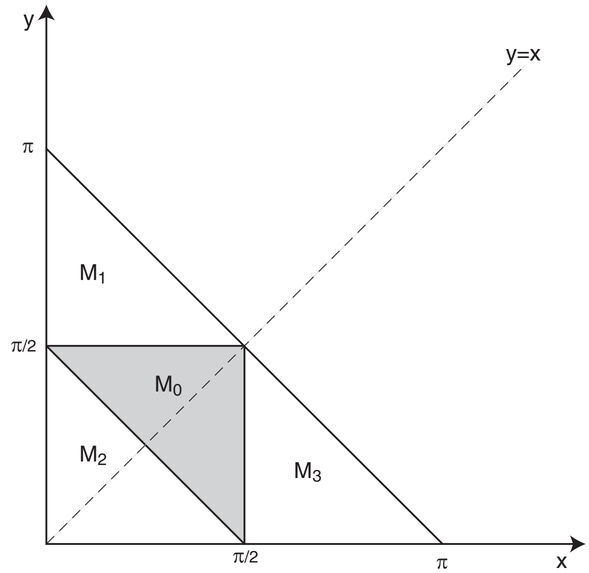

FIGURE 6. The Index Domain

Definition 3.1. A function $f : I \longrightarrow \mathbb{R}$ $(n > 1)$ is called *Schur-convex* if for every doubly stochastic matrix S,

$$f(S\mathbf{x}) \leq f(\mathbf{x}) \tag{2}$$

for all $\mathbf{x} \in I$. It is called *strictly Schur-convex* if the inequality is strict and *Schur-concave* (respectively, *strictly Schur-concave*) if the inequality (2) is reversed.

Next we will show that there is a two-parameter family of Sierpiński pedal triangles that can be indexed by points in I. Their area ratio is a Schur-concave function on I and their Hausdorff dimension can be described as a function on I as well. To this end, for $(x, y) \in I$, define

$$f(x, y) = -2\cos x \cdot \cos y \cdot \cos(x + y).$$

It is clear that f is symmetric and differentiable on I. Moreover

$$f_x = 2\sin x \cos y \cos(x + y) + 2\cos x \cos y \sin(x + y),$$

$$f_y = 2\cos x \sin y \cos(x + y) + 2\cos x \cos y \sin(x + y),$$

so

$$f_x - f_y = 2\cos(x + y) \cdot \sin(x - y).$$

Therefore, for any $(x, y) \in I$ with $x \neq y$ we always have

$$\left(f_x - f_y\right) \cdot (x - y) = 2[\cos(x + y)\sin(x - y)] \cdot (x - y) < 0$$

because $\cos(x + y) < 0$ and $[\sin(x - y)] \cdot (x - y) > 0$ on I. But $\left(f_x - f_y\right) \cdot (x - y) < 0$ for all $x \neq y$ is equivalent to f being a strict Schur-concave function on I. From the special properties of Schur-concave functions (see [MO79, RV73, Zha98]), if $f(x, y)$ has maximum in I, it can occur only in the subset of I where $x = y$. That is, $f(x, y)$ attains its maximum only along the

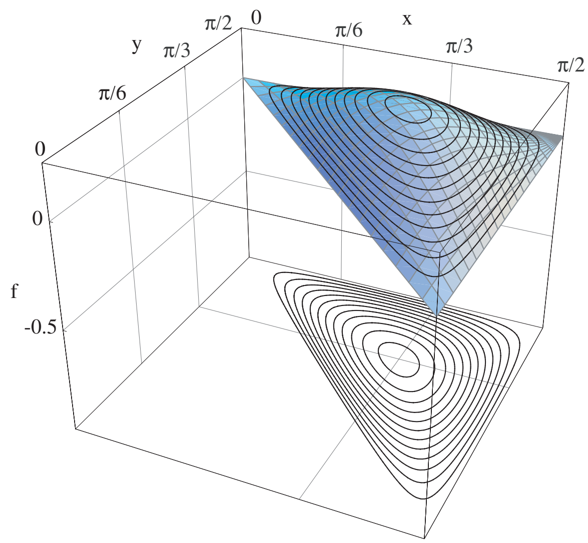

FIGURE 7. Area Ratios for Sierpiński Triangles

intersection of I with the main diagonal of the xy-plane. This can also be seen by directly setting $f_x = 0$ and $f_y = 0$ and solving the system simultaneously. added by RH

Now, consider f along the line $x = y$. It reduces to

$$f(x) = -2\cos^2 x \cos 2x = 2\cos^2 x - 4\cos^4 x.$$

It is a simple calculation in calculus to find that f has only one critical point $x_0 = \pi/3$, and f attains the maximum there.

We define an order "$\prec$" on I as follows.

Definition 3.2. We say $(x_1, y_1) \prec (x_2, y_2)$ if and only if there is a p, $0 < p < 1$, such that

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

That is, (x_2, y_2) can be obtained from (x_1, y_1) via multiplication by a 2×2 doubly stochastic matrix. Equivalently, (x_2, y_2) lies in the interior of the line segment between (x_1, y_1) and (y_1, x_1).

We can summarize the above arguments with the following theorem.

Theorem A. *Every similarity class of acute triangles can be represented by a point (x, y) in the index domain I. Furthermore,*

(i) $f(\frac{\pi}{3}, \frac{\pi}{3}) = \frac{1}{4}$ *is the maximum;*
(ii) $f(x, y) \to 0$ *as* $(x, y) \to \partial M_0$*;*
(iii) $f(x_1, y_1) < f(x_2, y_2)$ *if* $(x_1, y_1) \prec (x_2, y_2)$.

So given any acute triangle, the area ratio of its pedal triangle with the original triangle depends on the shape of the parental triangle. The closer the parental triangle is to being equilateral, the larger the area ratio. This is illustrated in Figure 7, which was generated with Mathematica.

4. Fractal dimension of SPT

In general, the computation of the fractal dimension of a set can be very complex and difficult. But for a self-similar fractal set, its fractal dimension can be calculated by the following useful formula [Bar89, Har89],

$$\sum_{i=1}^{n} r_i^D = 1,$$

where n is the number of the self-similar pieces reproduced in each step in the construction of a fractal, and the r_i's are the contraction ratios (or magnification factors) for $i = 1, 2, \cdots, n$ and D is its fractal dimension.

For the case of ST, $n = 3$, $r_1 = r_2 = r_3 = 1/2$, and the above equation easily implies that $D = \ln 3/\ln 2$. This is perhaps the simplest and most useful formula to find the Hausdorff dimensions of self-similar fractal sets with constant contraction ratios. However, for the SPT case, r_1, r_2 and r_3 are different, in general. Solving that algebraic equation for D as a function of r_1, r_2 and r_3 is not a simple task.

Again, let $\triangle ABC$ be an acute triangle and $\triangle A_1B_1C_1$ its pedal triangle. From formula (1b) we see that the three contraction ratios of the smaller triangles are $r_1 = \cos A$, $r_2 = \cos B$, and $r_3 = \cos C$. Therefore, the fractal dimension D of SPT associated with $\triangle ABC$ is determined by

$$\cos^D A + \cos^D B + \cos^D C = 1. \tag{3}$$

In particular, the fractal dimension of SPT depends on the initial triangle.

Example 4.1.

(i) If $A = B = C = \pi/3$, $\triangle ABC$ is an equilateral triangle, and the SPT is the same as the ST. So $D = \ln 3/\ln 2 \approx 1.58496$.

(ii) $A = \pi/3$, $B = \pi/4$, and $C = 5\pi/12$, we have

$$\left(\frac{1}{2}\right)^D + \left(\frac{\sqrt{2}}{2}\right)^D + \left(\cos\frac{5\pi}{12}\right)^D = 1.$$

Solving this numerically yields $D \approx 1.63343$.

(iii) $A = \pi/2$, $B = \pi/2 - C$, a right triangle, then equation 3 becomes

$$0^D + \cos^D B + \sin^D B = 1$$

which implies $D = 2$.

Let $D(x, y)$ denote the fractal dimension of the SPT generated by an acute triangle represented by a point (x, y) in the index domain I. We have

Theorem B. *$D(x, y)$ attains an absolute minimum value of $\frac{\ln 3}{\ln 2}$ on the index domain I at the point $(\pi/3, \pi/3)$.*

Proof. First we will show that $D(x, y)$ has a relative minimum at $(\pi/3, \pi/3)$. Rewriting equation 3 as

$$\cos^D x + \cos^D y + \cos^D(z) = 1$$

where $z = \pi - x - y$ and differentiating implicitly, we obtain

$$\begin{aligned}&\left[\cos^D x \cdot \ln(\cos x) + \cos^D y \cdot \ln(\cos y) + \cos^D z \cdot \ln(\cos z)\right] \cdot D_x = \\ &\quad \cos^D x \cdot D \cdot \tan x - \cos^D z \cdot D \cdot \tan z, \quad \text{and}\end{aligned} \tag{4}$$

$$\begin{aligned}(5)\qquad &\left[\cos^D x \cdot \ln(\cos x) + \cos^D y \cdot \ln(\cos y) + \cos^D z \cdot \ln(\cos z)\right] \cdot D_y = \\ &\cos^D y \cdot D \cdot \tan y - \cos^D z \cdot D \cdot \tan z.\end{aligned}$$

Since the coefficients of D_x and D_y in the above equations can never be zero for $(x, y) \in I$, we have $D_x = D_y = 0$ if and only if

$$\cos^D x \cdot D \cdot \tan x - \cos^D z \cdot D \cdot \tan z = 0, \quad \text{and}$$
$$\cos^D y \cdot D \cdot \tan y - \cos^D z \cdot D \cdot \tan z = 0.$$

An easy check shows that $D_x(\pi/3, \pi/3) = D_y(\pi/3, \pi/3) = 0$, i.e., $(\pi/3, \pi/3)$ is a critical point of $D(x, y)$ inside the index domain I. We can find the second order partial derivatives of D via implicit differentiation on equations 4 and 5. Through a lengthy but direct calculation, we find that at the point $(x, y) = (\pi/3, \pi/3)$,

$$\begin{vmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{vmatrix} = \begin{vmatrix} 2(\log_2 3 - 4/3) & \log_2 3 - 4/3 \\ \log_2 3 - 4/3 & 2(\log_2 3 - 4/3) \end{vmatrix} \left(\frac{\ln 3}{(\ln 2)^2}\right)^2 = 3\left(\log_2 3 - \frac{4}{3}\right)^2 \left[\frac{\ln 3}{(\ln 2)^2}\right]^2 > 0.$$

This shows that $(\pi/3, \pi/3)$ is a relative minimum of D on I.

To see that $(\pi/3, \pi/3)$ is the absolute minimum on I, we will show it is the only critical point on I. To do this, assume that (a, b) is a critical point on I with $D_x(a, b) = 0 = D_y(a, b)$. Then equations 4 and 5 imply $\cos^D a \cdot \tan a = \cos^D b \cdot \tan b$. Then

$$\left(\frac{\cos a}{\cos b}\right)^D = \frac{\tan b}{\tan a}.$$

Solving for D, and assuming $x \neq y$, gives

$$D(a, b) = \frac{\ln\left(\frac{\tan b}{\tan a}\right)}{\ln\left(\frac{\cos a}{\cos b}\right)}.$$

Next, we note that $D(x, y)$ possesses a six-fold symmetry over I. This is seen by the fact that the fractal dimension of a triangle with angles $(x, y, \pi - x - y)$ can be computed on I using any two of the three angles in either order: $D(x, y) = D(y, x) = D(x, \pi - x - y) = D(\pi - x - y, x) = D(y, \pi - x - y) = D(\pi - x - y, y)$.

STILL NEEDS WORK

This completes the proof of Theorem B. □

Let (x, y) be a point in the index domain I and let $D(x, y)$ be the fractal dimension of SPT generated by the triangle with inner angles x, y, and $\pi - (x + y)$. We have the following two conjectures.

Theorem C. *With notation as above,*

$$\frac{\ln 3}{\ln 2} \leq D(x, y) < 2.$$

Conjecture D. *With notation as above, $D(x_1, y_1) \leq D(x_2, y_2)$ if $(x_2, y_2) \prec (x_1, y_1)$ for any $(x_1, y_1), (x_2, y_2) \in I$.*

Remark 4.2.

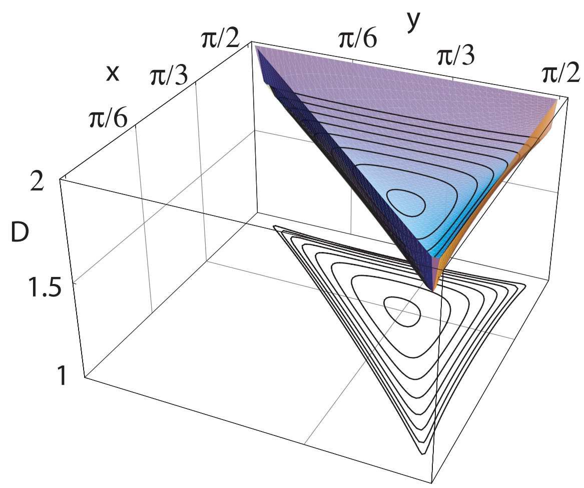

FIGURE 8. The Dimension, $D(x, y)$, of SPT on I

(i) Theorem C follows immediately from Theorem B since $D(\pi/3, \pi/3)$ is a global minimum.
(ii) Conjecture D claims that $D(x, y)$ is a Schur-convex function on I. The function $D(x, y)$ is illustrated in Figure 8.
(iii) Conjecture D is supported by the data shown in Figure 8, which was generated with Mathematica. That is, the more a triangle (x, y) deviates from the equilateral triangle $(\frac{\pi}{3}, \frac{\pi}{3})$, the bigger its fractal dimension $D(x, y)$. Some specific calculations are contained in Table 1.

Triangle	$D(x, y)$
$(25°, 75°, 80°)$	1.875
$(35°, 65°, 80°)$	1.6875
$(45°, 60°, 75°)$	1.6337
$(50°, 60°, 70°)$	1.6208
$(55°, 60°, 65°)$	1.611237
$(60°, 60°, 60°)$	1.584796

TABLE 1: Triangles And The Fractal Dimensions Of Their SPT

Figures 9, 10, 11, 12, 4, and 3 illustrate the Sierpiński pedal triangles corresponding, respectively, to the entries in the table. Entry six in the table is the Sierpiński triangle already shown in Figure 3. These images were produced with *Mathematica* using a modification of code for producing Sierpiński triangles found in [Wag00].
(iv) If the initial triangle $\triangle ABC$ is obtuse, then it generates a self-similar fractal SPT with overlaps. One may still talk about its fractal dimension, but more detailed discussion about Hausdorff measure theory is required. Interested readers may refer to [Mat95].

In summary, we may view SPT as a natural generalization of ST, and represent SPT as a two-parameter family of fractal sets constructed from triangles.

Moreover, for the roles played by ST in different circumstances such as in

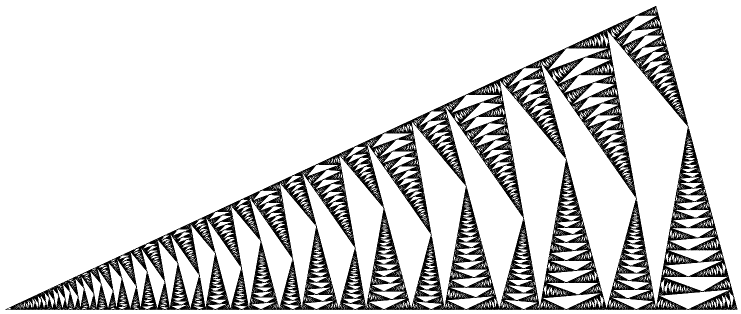

FIGURE 9. 25°-75°-80° Triangle, 50 Iterations

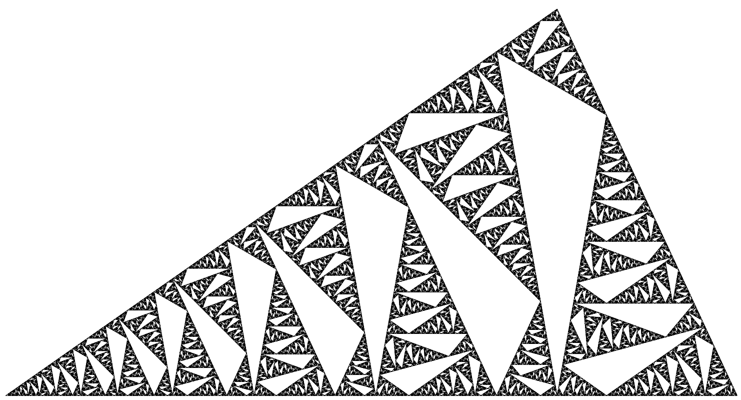

FIGURE 10. 35°-65°-80° Triangle, 30 Iterations

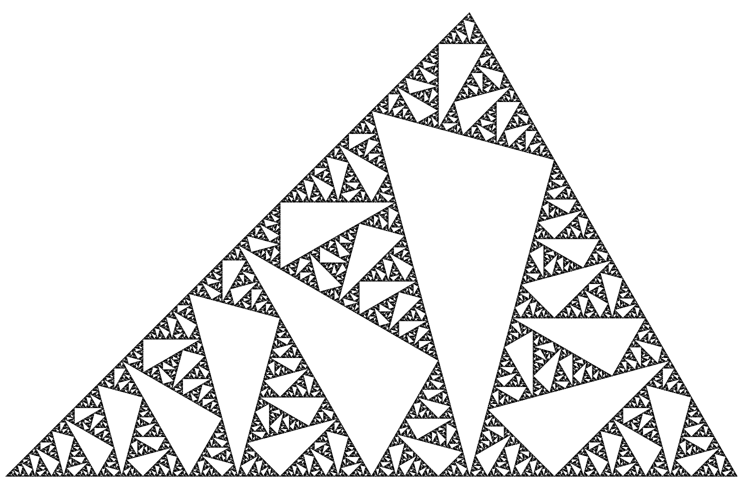

FIGURE 11. 45°-60°-75° Triangle, 20 Iterations

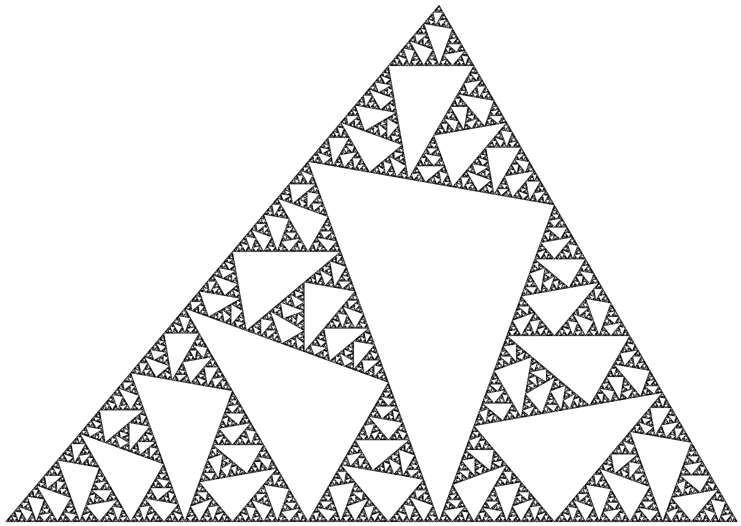

FIGURE 12. 50°-60°-70° Triangle, 15 Iterations

chaos games and in IFS [Bar89, Man82, Ste95], we hope that SPT can be introduced naturally into different areas of studies as a generalization of ST.

5. ITERATED FUNCTION SYSTEMS

Insert explanation for Figure 13. The dots are small for printing. You may need to zoom in to view them on a screen

6. SEQUENCES OF PEDAL TRIANGLES

this repeats the introduction

Another interesting property of pedal triangles is that they can form natural sequences in which the shapes change chaotically. Let T_0 be a triangle with inner angles A_0, B_0, and C_0. Construct a second triangle T_1 with inner angles A_1, B_1, and C_1 whose vertices are the feet of the three altitudes of T_0. Construct a third triangle T_2 with inner angles A_2, B_2, and C_2 whose vertices are the feet of the three altitudes of T_1. Construct a triangle T_3 from T_2 in the same way, and so on. One obtains a sequence of triangles $\{T_n\}_{n=0}^{\infty}$ where T_{n+1} is the pedal triangle of T_n. It is called the *sequence of pedal triangles* generated by T_0. See Figure 14.

reference?

While the size of these pedal triangles gets smaller rapidly, an interesting question is: "what can we say about the change of their shapes?" This problem and questions involving limiting shape of different sequences of plane triangles have been studied since at least a century ago [Hob97]. In the late 1980's, Kingston and Synge revisited the sequence of pedal triangles problem [KS88]. They discovered many interesting properties of such sequences and also fixed some errors that occurred in the earlier literature. Soon after their work, a number of articles made nice connections between the sequence of pedal triangles and symbolic dynamic systems and ergodic theory [Ale93, Hob97, Lax90, MO79, Ung90]. It seems that many fundamental concepts

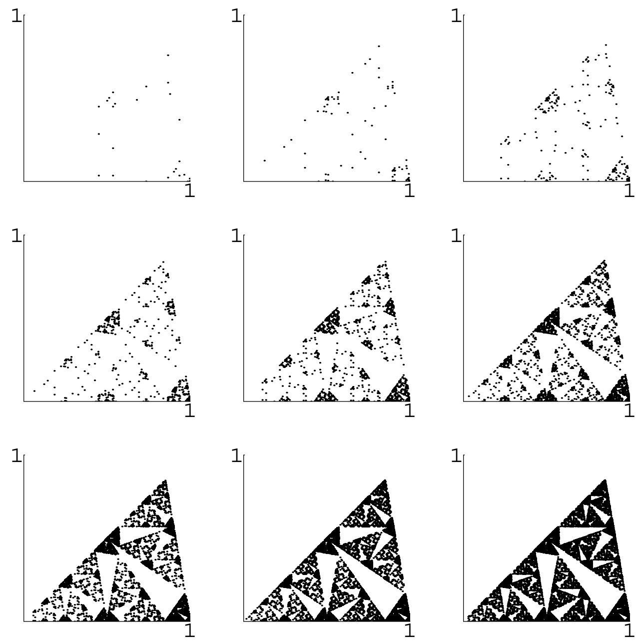

FIGURE 13. SPT 45°-55°-80° Triangle using IFS

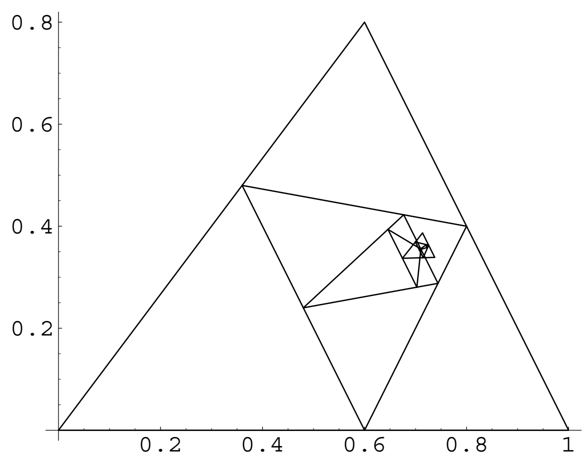

FIGURE 14. A Sequence of Pedal Triangles

and phenomenon in the study of dynamic systems can be found in sequence of pedal triangles. Alexander pointed out that one could use the special properties of these pedal triangles as an elementary and expository introduction to the power and the beauty of symbolic dynamic systems [Ale93]. Additionally, the well-established theory of symbolic dynamic systems, ergodic theory, fractal geometry, and computer graphics will no doubt provide powerful new tools for many classical geometry problems that have proven difficult to tackle using the classical synthetic methods.

In [DHZ03, HZ01], we investigated some interesting dynamic systems problems in classical geometry. Different sequences of triangles always provided important and inspiring examples. In particular, the chaotic behavior of the sequence of pedal triangles motivate people to further explore the possible intrinsic connections between classical geometry problems and some advanced mathematical theories. It is interesting to note that when a dynamic system proved to be chaotic, fractal appears in one way or another although the two subjects are completely independent from each other. Most textbooks of dynamic systems treat fractals as the strange attractors, but fractals occur everywhere.

To conclude this article, we would like to pose: Is the fractal dimension of a Sierpiński pedal triangle generated by a given triangle T_0 related to chaotic property of the sequence of pedal triangles generated by T_0?

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